Turbulent Two-Dimensional Jet

Graduate Students: R.H. Bhana, L. Lekoko, B. Mugwangwavari
Supervisors: Prof. D.P. Mason, E. Mubai, K. Born

MISGSA2021

Turbulent Flow

- Turbulent Flow: Irregular or random fluctuation (mixing or rotational motion) that is superimposed on the mainstream
- Feature of of fluid flow and not of a fluid
- Turbulent flow = mean flow + random fluctuations with zero mean

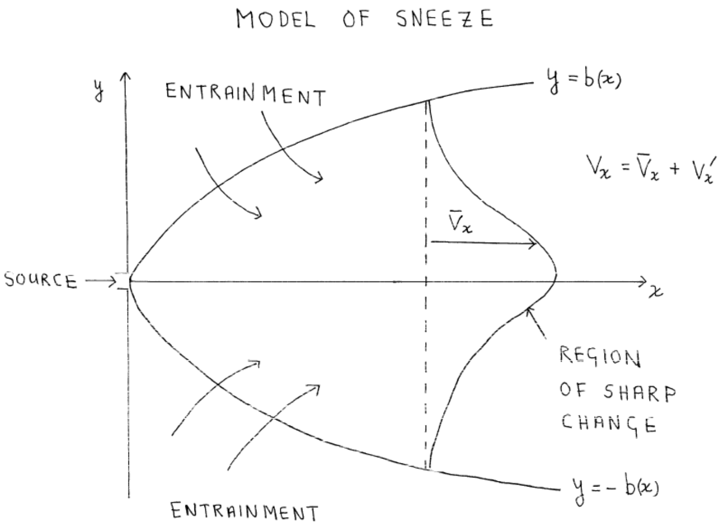
$$V_i(x,y,t) = \overline{V}_i(x,y) + V_i^{'}(x,y,t) \tag{1}$$

$$\overline{\overline{V}}_i = \overline{V}_i, \overline{V_i^{'}} = 0, \overline{V_i^{'} V_j^{'}} \neq 0$$

Aims and Objectives

- Model of sneeze
- Fit parameter values for a sneeze
- Range of jet in downstream x-direction
- Range of jet in transverse y-direction
- Estimates of Social Distancing
- Model effect of mask by changing value of J

Model of a Sneeze



Governing Equations

- Reynolds averaged equations in the boundary layer approximation:

$$\overline{V}_x \frac{\partial \overline{V}_x}{\partial x} + \overline{V}_y \frac{\partial \overline{V}_x}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \nu_T\right)\frac{\partial \overline{V}_x}{\partial y}\right] \tag{2}$$

$$\nu_T = \ell^2(x)\left|\frac{\partial \overline{V}_x}{\partial y}\right| \tag{3}$$

- Conservation of Mass:

$$\frac{\partial \overline{V}_x}{\partial x} + \frac{\partial \overline{V}_y}{\partial y} = 0 \tag{4}$$

Two-Dimensional Boundary Layer Equation

- The two-dimensional boundary equation is:

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial}{\partial y}\left[\left(\nu - \ell^2(x)\frac{\partial^2 \Psi}{\partial y^2} \right) \frac{\partial^2 \Psi}{\partial y^2} \right], \tag{5}$$

$$\overline{V}_x = \frac{\partial \Psi}{\partial y}, \; \overline{V}_y = -\frac{\partial \Psi}{\partial x}$$

Proving Invariance

- Scaling transformations:

$$\bar{x} = \lambda^a x,\ \bar{y} = \lambda^b y,\ \overline{\Psi} = \lambda^c \Psi,\ \bar{\ell} = \lambda^m \ell \tag{6}$$

- Equation (5) is invariant,

$$\frac{\partial \overline{\Psi}}{\partial \bar{y}} \frac{\partial^2 \overline{\Psi}}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \overline{\Psi}}{\partial \bar{x}} \frac{\partial^2 \overline{\Psi}}{\partial \bar{y}^2} = \frac{\partial}{\partial \bar{y}} \left[\left(\nu - \bar{\ell}^2(\bar{x}) \frac{\partial^2 \overline{\Psi}}{\partial \bar{y}^2} \right) \frac{\partial^2 \overline{\Psi}}{\partial \bar{y}^2} \right] \tag{7}$$

- Provided

$$c = a - b,\ m = \frac{1}{2}\left(3b - a\right) \tag{8}$$

General Solution of Streamline Function

- Suppose (5) has a general solution:

$$\Psi = f(x, y) \tag{9}$$

- With corresponding invariant solution:

$$\overline{\Psi} = f(\overline{x}, \overline{y}) \tag{10}$$

- Streamline function:

$$\Psi(x, y) = x^{\beta} F(\xi),\ \beta = \frac{c}{a} \tag{11}$$

$$\xi = y x^{-\alpha},\ \alpha = \frac{b}{a} \tag{12}$$

General solution of Prandtl's Mixing Length

- Suppose a solution of the form:

$$\ell = h(x) \tag{13}$$

- With corresponding invariant solution:

$$\bar{\ell} = h(\bar{x}) \tag{14}$$

- Prandtl's mixing length:

$$\ell(x) = \ell_0 x^{\frac{m}{a}} \tag{15}$$

- Note that ℓ_0 is a constant

Momentum Flux

- Given by

$$J = 2\rho \int_0^{b(x)} \overline{V}_x^2(x,y)dy \tag{16}$$

$$= 2\rho \int_0^{b(x)} \left(\frac{\partial \Psi}{\partial y}\right)^2 dy \tag{17}$$

- Since $J = \textit{constant independent of } x$:

$$\alpha = 2\beta, \tag{18}$$

$$b(x) = \xi_b\, x^{\alpha} \tag{19}$$

- Note that ξ_b is a constant

Case 1: $\nu \neq 0$, $\alpha = \frac{2}{3}$

- Three unknowns, two equations
- Parameter values:

$$\alpha = \frac{2}{3},\ \beta = \frac{1}{3},\ \frac{m}{a} = \frac{1}{2} \tag{20}$$

- Hence,

$$\Psi(x,y) = x^{\frac{1}{3}} F(\xi),\ \xi = yx^{-\frac{2}{3}},\ \ell(x) = \ell_0\, x^{\frac{1}{2}},\ b(x) = \xi_b\, x^{\frac{2}{3}} \tag{21}$$

- Prandtl's hypothesis is not satisfied

Case 2: $\nu = 0$, $\alpha = 1$

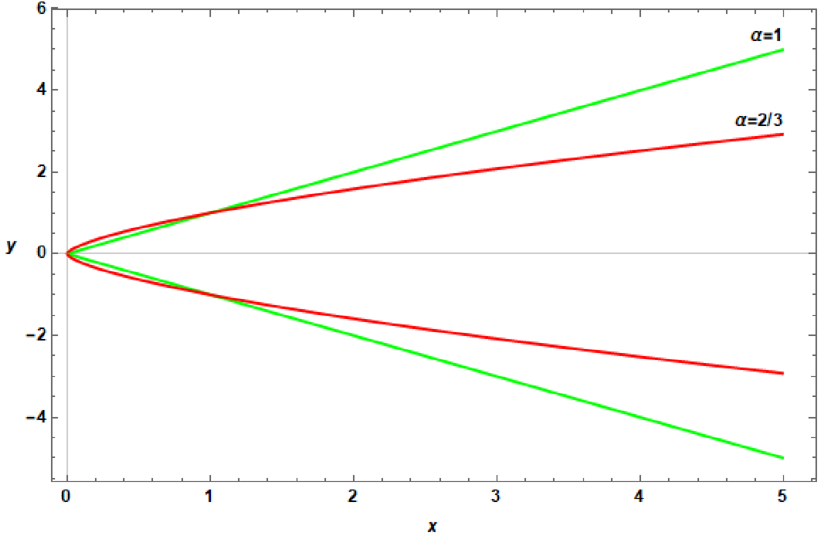
- Three unknowns, one equation
- Satisfying Prandtl's hypothesis

$$\alpha = 1 \tag{22}$$

- Hence,

$$\Psi(x,y) = x^{\frac{1}{2}} F(\xi),\ \xi = yx^{-1},\ \ell(x) = \ell_0\, x,\ b(x) = \xi_b\, x \tag{23}$$

Plot of the Boundary Equation b(x) for Different Parameters of α



Velocities

- ODE must be independent of x and y. Rewrite similarity solution i.t.o ξ and x

$$y = \xi x^{\alpha} \tag{24}$$

$$\begin{aligned} V_x(x,y) &= \frac{\partial \Psi}{\partial y} \\ &= x^{\frac{-\alpha}{2}} \frac{dF}{d\xi} \end{aligned} \tag{25}$$

$$\begin{aligned} V_y(x,y) &= -\frac{\partial \Psi}{\partial x} \\ &= \frac{\alpha}{2} x^{\frac{\alpha}{2}-1} \left[-F(\xi) + 2\xi \frac{dF}{d\xi} \right] \end{aligned} \tag{26}$$

Boundary Conditions

$$\frac{\partial \overline{V}_x}{\partial y}(x,0) = 0 \; (\textit{Turning point}) \tag{27}$$

$$\overline{V}_y(x,0) = 0 \; (\textit{No cavities}) \tag{28}$$

$$\frac{\partial \overline{V}_x}{\partial y}(x,b(x)) = 0 \; (\textit{Zero kinematic eddy viscosity}) \tag{29}$$

$$\overline{V}_x(x,b(x)) = 0 \; (\textit{Limit tends to zero}) \tag{30}$$

Ordinary Differential Equations (ODEs)

- $\nu \neq 0$, $\alpha = \frac{2}{3}$:

$$\ell_0^2 \left(\frac{d^2 F}{d\xi^2} \right)^2 - \nu \frac{d^2 F}{d\xi^2} - \frac{1}{3} F \frac{dF}{d\xi} = 0 \tag{31}$$

- $\nu = 0$, α is arbitrary

$$\ell_0^2 \left(\frac{d^2 F}{d\xi^2} \right)^2 - \frac{\alpha}{2} F \frac{dF}{d\xi} = 0 \tag{32}$$

Analytical Solution for Case 2

- Equation (32) is solved analytically in order to determine the parameters for the velocity equations
- Find values for the constants, ξ_b, and K

Entrainment

- At the boundary $y = b(x) = \xi_b x^\alpha$:

$$\xi_b = \frac{4\pi}{3\sqrt{3}} \left(\frac{2\ell_0^2}{\alpha}\right)^{\frac{1}{3}} \tag{33}$$

Boudary of a Jet

- Determining the Value for K:

$$K = \left(\frac{J}{2\rho}\right)^{\frac{1}{2}} \left(\frac{2\ell_0^2}{\alpha}\right)^{\frac{1}{6}} \frac{\left[\Gamma\left(\frac{1}{3}\right)\right]^{\frac{1}{2}}}{\Gamma\left(\frac{2}{3}\right)} \tag{34}$$

- Entrainment:

$$v_y(x, b(x)) = -\frac{\alpha}{2} x^{\left(\frac{\alpha}{2} - 1\right)} K \tag{35}$$

Velocity in the x-direction

- At the centerline of the Jet $y = 0$:
- Velocity

$$v_x(x,0) = \left(\frac{\alpha}{2\ell_0^2}\right)^{\frac{1}{3}} K x^{-\frac{\alpha}{2}} \tag{36}$$

Numerical Solution

- Equation (31) is solved numerically
- Quadratic form and making the second derivative the subject of the formula
- Shooting method
- Shooting towards a conserved quantity and not a derivative

Expectation